Tutorial 8 (Mar 12, 14)

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Q1) (Midterm Q5) Let $f:[0,1]\times[0,1]\to\mathbb{R}$ be defined as

$$f(x,y)=\begin{cases}1, & x: \text{rational and } 0\le y<\frac{1}{2}\\ 0, & x: \text{rational and } \frac{1}{2}\le y<1\\ 0, & x: \text{irrational and } 0\le y<\frac{1}{2}\\ 1, & x: \text{irrational and } \frac{1}{2}\le y<1\end{cases}$$

(a) Show that $\int_0^1\int_0^1 f(x,y)\,dx\,dy$ does not exist.

(b) Show that $\int_0^1\int_0^1 f(x,y)\,dy\,dx$ exists and find its value.

Sol) (a) It suffices to show that $\forall\, y\in[0,1]$, $\int_0^1 f(x,y)\,dx$ does not exist.

Case 1: $0\le y<\frac{1}{2}$, then $f(x,y)=\begin{cases}1, & x: \text{rational}\\ 0, & x: \text{irrational}\end{cases}$

$\therefore$ For any partition $P=\{I_k\}_{k=1}^n$, $\forall\, 1\le k\le n$, $\exists\, x_k, \tilde{x}_k\in I_k$ s.t.

x_k: rational and $\tilde{x}_k$: irrational. $\therefore f(x_k,y)=1$; $f(\tilde{x}_k,y)=0$

$$\therefore \lim_{|P|\to 0}\sum_{k=1}^n f(x_k,y)\Delta x_k = 1\neq 0=\lim_{|P|\to 0}\sum_{k=1}^n f(\tilde{x}_k,y)\Delta x_k$$

Case 2: $\frac{1}{2}<y\le 1$. Same notations as above, but $f(x_k,y)=0$; $f(\tilde{x}_k,y)=1$

$$\lim_{|P|\to 0}\sum_{k=1}^n f(x_k,y)\Delta x_k = 0\neq 1=\lim_{|P|\to 0}\sum_{k=1}^n f(\tilde{x}_k,y)\Delta x_k$$

$\therefore$ In both cases, $\int_0^1 f(x,y)\,dx$ does not exist.

(b) Step 1: Show that $\forall x \in [0,1]$, $\int_0^1 f(x,y)\,dy$ exists and find its value.

Case 1: x: rational, then $f(x,y) = \begin{cases} 1, & 0 \le y < \frac{1}{2} \\ 0, & \frac{1}{2} \le y \le 1 \end{cases}$

$\therefore$ $f(x,y)$ is continuous in y except $y = \frac{1}{2}$, and hence is integrable.

and $\int_0^1 f(x,y)\,dy = \int_0^{\frac{1}{2}} f(x,y)\,dy + \int_{\frac{1}{2}}^1 f(x,y)\,dy = [y]_0^{\frac{1}{2}} + 0 = \frac{1}{2}$

Case 2: x: irrational, then $f(x,y) = \begin{cases} 0, & 0 \le y < \frac{1}{2} \\ 1, & \frac{1}{2} \le y \le 1 \end{cases}$

$\therefore$ $f(x,y)$ is continuous in y except $y = \frac{1}{2}$, and hence is integrable.

and $\int_0^1 f(x,y)\,dy = \int_0^{\frac{1}{2}} f(x,y)\,dy + \int_{\frac{1}{2}}^1 f(x,y)\,dy = 0 + [y]_{\frac{1}{2}}^1 = \frac{1}{2}$

$\therefore$ In both cases, $\int_0^1 f(x,y)\,dy$ exists and $= \frac{1}{2}$.

Step 2: Show that $\int_0^1 \int_0^1 f(x,y)\,dy\,dx$ exists and find its value.

$\int_0^1 \int_0^1 f(x,y)\,dy\,dx = \int_0^1 \frac{1}{2}\,dx$ exists

$= \left[\frac{x}{2}\right]_0^1 = \frac{1}{2}$ //

Q2) (Midterm Q4)

Evaluate $\iiint_D |xyz|\,dV$, where

(a) D is the unit ball in $\mathbb{R}^3$.

(b) D is the solid enclosed by the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ where $a,b,c>0$

Sol) (a) Step 1: Simplify the integral by the symmetry.

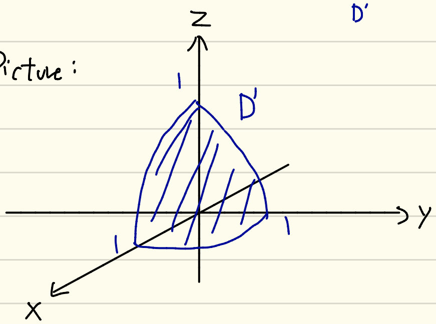
Note that the function $f(x,y,z)=|xyz|$ is even w.r.t. x,y,z
(i.e. $f(\pm x,\pm y,\pm z)=f(x,y,z)$)

and D is symmetric w.r.t. x,y,z (i.e. $(x,y,z)\in D \Leftrightarrow (\pm x,\pm y,\pm z)\in D$)

$\therefore \iiint_D |xyz|\,dV = 8\cdot\iiint_{D'} |xyz|\,dV$, where D' is the portion of unit ball that lies in the first octant.

$$= 8\cdot\iiint_{D'} xyz\,dV$$

Picture:



Step 2: Describe D' using spherical coordinates.

$$D' = \{(\rho, \phi, \theta) \in [0,+\infty) \times [0,\pi] \times [0,2\pi) \mid 0 \le \rho \le 1;\ 0 \le \phi \le \tfrac{\pi}{2};\ 0 \le \theta \le \tfrac{\pi}{2}\}$$

Step 3: Compute $\iiint_{D'} xyz\, dV$ using spherical coordinates.

$$\iiint_{D'} xyz\, dV = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 (\rho \sin\phi \cos\theta)(\rho \sin\phi \sin\theta)(\rho\cos\phi)(\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta)$$

$$= \left(\int_0^{\frac{\pi}{2}} \sin\theta \cos\theta\, d\theta\right)\left(\int_0^{\frac{\pi}{2}} \sin^3\phi \cos\phi\, d\phi\right)\left(\int_0^1 \rho^5\, d\rho\right)$$

$$= \left[\frac{\sin^2\theta}{2}\right]_0^{\frac{\pi}{2}} \left[\frac{\sin^4\phi}{4}\right]_0^{\frac{\pi}{2}} \left[\frac{\rho^6}{6}\right]_0^1 = \frac{1}{48}$$

Step 4: Compute $\iiint_D |xyz|\, dV$.

$$\iiint_D |xyz|\, dV = 8 \iiint_{D'} xyz\, dV = 8 \cdot \frac{1}{48} = \frac{1}{6}$$

(b) Step 1: Apply a change of variable $\begin{cases} x = au \\ y = bv \\ z = cw \end{cases}$.

then $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \iff u^2+v^2+w^2 = 1$

$\therefore$ D is transformed to the unit ball D'' in $\mathbb{R}^3$.

Step 2: Compute the Jacobian $\frac{\partial(x,y,z)}{\partial(u,v,w)}$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

Step 3: Apply the change of variables formula.

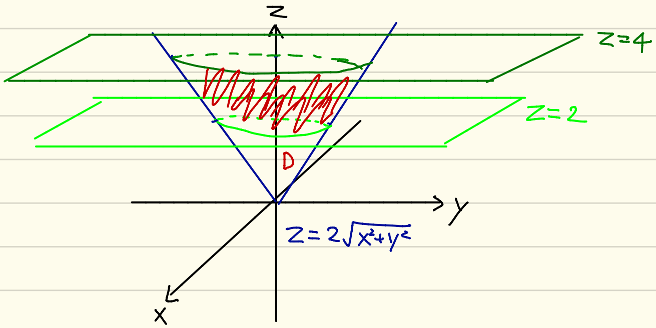
$$\iiint_D |xyz| \, dV = \iiint_{D''} |au \cdot bv \cdot cw| \, (|abc| \, du\,dv\,dw)$$

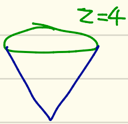
$$= a^2b^2c^2 \iiint_{D''} |uvw| \, dV$$

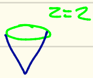
$$= a^2b^2c^2 \cdot \frac{1}{6} \quad \text{(by (a))}$$

Q3) (Midterm Q3) Find the volume of the solid bounded by the cone $z = 2\sqrt{x^2+y^2}$ between the planes $z=2$ and $z=4$.

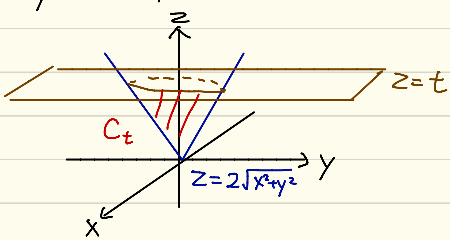
Sol) Step 1: Sketch the solid D.

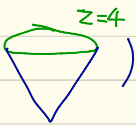


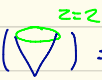
Step 2: Compute the volumes of 

and 

More generally: Compute the volume of C_t, where $t>0$



then $\mathrm{Vol}(D) = \mathrm{Vol}($ 

$) - \mathrm{Vol}($ 

$) = \mathrm{Vol}(C_4) - \mathrm{Vol}(C_2)$

Step 3 : Describe C_t using cylindrical coordinates.

Note that $z = 2\sqrt{x^2+y^2} \iff z = 2r \iff r = \frac{z}{2}$

$$\therefore C_t = \{(r,\theta,z) \in [0,+\infty) \times [0,2\pi) \times [0,t] \mid 0 \le z \le t ; 0 \le \theta < 2\pi ; 0 \le r \le \tfrac{z}{2}\}$$

Step 4: Compute $Vol(C_t)$ using cylindrical coordinates.

$$Vol(C_t) = \int_0^t \int_0^{2\pi} \int_0^{\frac{z}{2}} r\,dr\,d\theta\,dz$$

$$= \left(\int_0^{2\pi} d\theta\right) \int_0^t \left[\frac{r^2}{2}\right]_0^{\frac{z}{2}} dz$$

$$= 2\pi \cdot \int_0^t \frac{z^2}{8}\,dz = \frac{\pi}{4}\left[\frac{z^3}{3}\right]_0^t = \frac{\pi}{12}t^3$$

Step 5: Compute $Vol(D)$.

$$Vol(D) = Vol(C_4) - Vol(C_2) = \frac{\pi}{12}(4^3 - 2^3) = \frac{14}{3}\pi \,/\!/$$